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Problem of Classical and Nonclassical Probabilities

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A characterization of classical and nonclassical probabilities expressed in terms of some inequalities between multidimensional or S-probability is given. A new criterion (not referring to correlation probabilities) for nonclassicality of the range of a complete S-probability measure on an event system is proposed.

1. INTRODUCTION

In approaches to the foundations of quantum mechanics the notion of event is often taken as a primitive ingredient, and the logical structure of events is given from the outset, or assumed axiomatically (this is the case in defining a propositional system in quantum logic). However, it appears closer to experience to focus attention on the measured probabilities of given observations in various states of the physical system and then to derive from these probabilities the logical structure of events which appear in the description of the physical system under consideration.

This last attitude raises the problem of deciding whether the empirically obtained probabilities have a classical or a nonclassical nature; more specifically, whether they are contained in the range of a classical or a nonclassical probability measure. This is the problem considered in this paper: we shall provide some new criteria about the classicality or nonclassicality of a set of probabilities. These criteria are an addition to what has been found by Accardi (1983, 1986), Gudder and Zanghì (1984), and Pitowsky (1989).

We shall provide a characterization of probabilities in terms of the multidimensional probability, or S-probability (physically interpreted as a family of probabilities labeled by the states of the physical system), a notion which seems to be especially suitable for expressing the properties which we

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are calling classical or nonclassical. We shall omit the proofs of our results: for them we refer to another recent paper (Beltrametti and Mączyński, 1991).

2. DEFINITIONS

We need first some definitions.

Let S be a nonempty set to be physically interpreted as the set of states of the physical system. By an S-indexed probability, briefly S-probability, we understand any function $p: S \to [0, 1]$, i.e., any $p \in [0, 1]^S$. If $S = \{\alpha\}$ is a one-element set, we call an S-probability simply a probability. If $S = \{\alpha_1, \ldots, \alpha_n\}$ is an n-element set, then an S-probability can be identified with a sequence of values $(p^{(1)}, \ldots, p^{(n)})$, i.e., with a vector in the n-dimensional space R^n. Hence p can be viewed as an n-dimensional probability or n-valued probability. We denote the set of all S-probabilities by $P(S)$, i.e., $P(S) = [0, 1]^S$.

Let $p, q \in P(S)$. Since p and q are real-valued functions, we can define a partial order relation in $P(S)$ by $p \leq q$ if and only if $p(x) \leq q(x)$ for all $x \in S$. By 0 and 1 we denote (with common abuse of notations) the functions on S which take the values 0 and 1 only, respectively. If $p, q \in P(S)$, then $p + q$ is always defined as the sum of functions p and q, although it may not belong to $P(S)$. If $p + q \in P(S)$, i.e., if $p + q \leq 1$, then we say that p and q are orthogonal. A triple of S-probabilities p_1, p_2, p_3 is said to be a triangle, denoted by $\Delta(p_1, p_2, p_3)$, if $p_i + p_j \leq 1$ for $i \neq j$, $i, j = 1, 2, 3$. If p is an S-probability, then $1 - p$ is always an S-probability, too. We write $p' = 1 - p$.

By an event system L we understand a triple $\mathrm{L} = (L, \leq, ')$, where L is a set, $\leq$ a partial order in L, $'$ a mapping $a \to a'$ from L into L such that $(L, \leq, ')$ is an orthomodular, orthocomplemented, partially ordered set (an orthoposet). The elements of L are called events.

An orthoposet $(L, \leq, ')$ satisfies the following set of axioms:

(i) $a'' = a$ for all $a \in L$
(ii) $a \leq b$ implies $b' \leq a'$ for all $a, b \in L$
(iii) if $a_1, a_2, \ldots, a_n$ is a sequence of members of L for which $a_i \leq a_j'$ for all $i \neq j$, then the least upper bound $a_1 \vee a_2 \vee \cdots \vee a_n$ exists in $(L, \leq)$
(iv) $a \vee a' = b \vee b'$ for all $a, b \in L$ ($a \vee a'$ will be denoted by **1**)
(v) $a \leq b$ implies $b = a \vee (a \vee b')'$

By a classical event system we understand an event system L which is a Boolean algebra with respect to the order $\leq$ and complementation $'$ (i.e., it is a complemented distributive lattice).

Let $\mathrm{L}=(L, \leq, ')$ be an event system, S a nonempty set. By an S-probability measure on L (or briefly on L) we understand any map

$$p: \quad L \to P(S)=[0, 1]^S$$

with the properties

$$p(\mathbf{0})=0, \qquad p(\mathbf{1})=1$$

$$p(a')=1-p(a) \qquad \text{for all} \quad a \in L$$

$$p(a_1 \vee a_2 \vee \cdots \vee a_n)=p(a_1)+p(a_2)+\cdots+p(a_n)$$

$$\text{whenever} \quad a_i \leq a_j' \text{ for } i \neq j$$

Hence, for every $a \in L$, $p(a)$ is an S-probability, i.e., a function from S into [0, 1], and $p(a)(\alpha)$ is a probability, i.e., a number between 0 and 1. We write $p^\alpha(a)$ instead of $p(a)(\alpha)$. We have as a corollary that for every $a \in S$ the map $a \to p^\alpha(a)$ is a probability measure on L.

By an event S-probability space we understand a pair (L, p) where L is an event system and p an S-probability measure on L.

Let p be an S-probability measure on L. We say that p is complete if $a \leq b \Leftrightarrow p(a) \leq p(b)$ for all $a, b \in L$.

Let K be a set of S-probabilities. We say that K is representable if there is an event S-probability space (L, p) such that

$$K \subseteq \{p(a): a \in L\}$$

We denote this embedding mapping by φ. We call (L, p, φ) a representation for K.

Observe that if (L, p, φ) is a representation for K, then $a \leq b$ in K always implies $\varphi(a) \leq \varphi(b)$ in $P(S)$. Similarly, $a+b=c$ implies $\varphi(a)+\varphi(b)=\varphi(c)$. Hence the embedding preserves all existing relations arising from the order and the complementation structure in K. In other words, the embedding is without any additional assumption an orthoposet isomorphism. This shows that S-probabilities intrinsically reflect the structure of relations between objects described by them. The structure of order and complementation is naturally contained in any set of S-probabilities.

We say that a set of probabilities K is classically representable if there is a representation (L, p, φ) for K such that L is a classical event system.

By a correlation sequence we shall understand an indexed sequence of S-probabilities $K=(p_1, p_2, \ldots, p_n, \ldots, p_{ij}, \ldots)$ where $1 \leq i, j \leq n$, $i<j$ (not all pairs i, j, $i<j$, need appear). We shall say that this sequence is consistently

representable if there is a representation (L, p, φ) for K and a sequence of pairwise compatible events $(a_1, \ldots, a_n)$ in L such that

$$p_i = p(a_i) \qquad \text{for} \quad i = 1, 2, \ldots, n$$

$$p_{ij} = p(a_i \wedge a_j)$$

whenever the pair i, j appears in K.

3. MATHEMATICAL RESULTS

If we consider the simplest correlation sequence (p_1, p_2, p_{12}) of S-probabilities we have the full equivalence of the following four conditions:

(I) K is consistently representable.

(II) The following inequalities hold:

$$0 \leq p_{12} \leq p_1 \leq 1, \qquad 0 \leq p_{12} \leq p_2 \leq 1, \qquad p_1 + p_2 - p_{12} \leq 1$$

(III) There exist $q_1, q_2, q_3 \in P(S)$, $q_1 + q_2 + q_3 \leq 1$, such that

$$p_1 = q_1 + q_3, \qquad p_2 = q_2 + q_3, \qquad p_{12} = q_3$$

(IV) K is consistently representable in a classical event space.

The proof of this fact is essentially contained in Pitowsky (1989, Theorem 2.3), or, more completely, in Beltrametti and Mączyński (1991, Theorem 1).

Thus we see that if we consider only two probabilities p_1 and p_2 with their correlation p_{12} we have no criterion to distinguish whether they come from a classical or a nonclassical situation: if the sequence (p_1, p_2, p_{12}) is consistently representable, then it is always classically representable.

The possibility of distinguishing between the classical and the nonclassical case comes only if we consider at least three probabilities p_1, p_2, p_3 with their correlations. For a sequence $(p_1, p_2, p_3, p_{12}, p_{13}, p_{23})$ of S-probabilities we have indeed that the following conditions are equivalent:

(I) K is classically representable.

(II) The following inequalities hold:

$$0 \leq p_{ij} \leq p_i \leq 1, \qquad 0 \leq p_{ij} \leq p_j \leq 1, \qquad 1 \leq i < j \leq 3$$

$$p_i + p_j - p_{ij} \leq 1, \qquad 1 \leq i < j \leq 3$$

$$p_1 - p_{12} - p_{13} + p_{23} \geq 0$$

$$p_2 - p_{12} - p_{23} + p_{13} \geq 0$$

$$p_3 - p_{13} - p_{23} + p_{12} \geq 0$$

$$p_1 + p_2 + p_3 - p_{12} - p_{13} - p_{23} \leq 1$$

(III) There exist $q_1, q_2, q_3, q_4, q_5, q_6, q_7 \in P(S)$, $\sum_{i=1}^{7} q_i \leq 1$, such that

$$p_1 = q_1 + q_2 + q_4 + q_7$$
$$p_2 = q_2 + q_5 + q_6 + q_7$$
$$p_3 = q_3 + q_4 + q_6 + q_7$$
$$p_{12} = q_2 + q_7$$
$$p_{13} = q_4 + q_7$$
$$p_{23} = q_6 + q_7$$

The proof of this fact is also essentially contained in Pitowsky (1989, Theorems 2.3 and 2.4), or, more completely, in Beltrametti and Mączyński, (1991, Theorem 2).

Notice that the last four inequalities in (II) are Bell's inequalities. Thus we have that if for a correlation sequence $(p_1, p_2, p_3, p_{12}, p_{13}, p_{23})$ Bell's inequalities are violated, then the sequence is not classically representable. As an example, take the sequence of probabilities (1/2, 1/2, 1/2, 1/8, 1/8, 1/8): the last of Bell's inequalities in (II) does not hold, so that this probability sequence does not admit a representation in a classical event space; it admits, however, a Hilbert space representation and it can be physically related to a system formed by a pair of spin-1/2 particles (Pitowsky, 1989).

The above criterion for classical or nonclassical representability cannot be directly generalized to larger sequences of S-probabilities (Pitowsky, 1989). Therefore, to deal with such a generalization one has to take another way. The following two theorems, whose proof is given in Beltrametti and Mączyński (1991, Theorems 3 and 4), provide an answer to the problem.

Theorem 1. Let K be a set of S-probabilities. Then the following conditions are equivalent:

(I) K is the range of a complete S-probability measure on some event system L.

(II) K has the properties

(i) $0 \in K$
(ii) $p \in K \Rightarrow 1 - p \in K$
(iii) $\triangle(p_1, p_2, p_3) \in K \Rightarrow p_1 + p_2 + p_3 \in K$

i.e., K contains 0 and is closed with respect to subtraction from 1 and sums of triangles.

Theorem 2. Let K be a set of S-probabilities. Then the following conditions are equivalent:

(I) K is the range of a complete S-probability measure on some classical event system L.

(II) K has the properties

(i) $0 \in K$
(ii) $p \in K \Rightarrow 1-p \in K$
(iii) $\triangle(p_1, p_2, p_3) \in K \Rightarrow p_1+p_2+p_3 \in K$
(iv) for any $p_1, p_2 \in K$ there exists a triangle $\triangle(q_1, q_2, q_3)$ in K such that $p_1=q_1+q_2$ and $p_2=q_2+q_3$.

While Theorem 1 provides us just a criterion to verify whether a set of S-probabilities is the range of a complete probability measure on an arbitrary (possibly nonclassical) event system, Theorem 2 gives the key to verify whether the underlying event system is classical or not.

As a corollary [whose proof is again contained in Beltrametti and Mączyński (1991)], we have the following statement that applies to any S-probability sequence K which is the range of a complete S-probability measure on some event system L:

K is nonclassical if and only if there exists a pair (p_1, p_2) of members of K such that whenever $q_1 \leq q_2$ and $p_1 \leq q_1+q_2$ for some $q_1, q_2 \in K$, then $p_1-p_2 \neq q_1+q_2-1$.

Such a criterion can be easily handled and has the advantage of being easily programmed for a computing procedure.

4. EXAMPLES

As anticipated, an S-probability will be physically interpreted as a set of probabilities (of occurrence of some experimental result) corresponding to different states of the physical system under consideration, and S will be understood as the set formed by these states.

The simplest example can be obtained assuming that S contains just one state α (so that S-probabilities are simply numbers in [0, 1]) and that a dichotomic observation is made, getting, say, $K=\{0, 1\}$. This is clearly a classical system, isomorphic to the two-element Boolean algebra.

Still assuming $S=\{\alpha\}$, suppose that a three-valued observation is made with $K=\{0, 1/2, 1\}$. This sequence of S-probabilities cannot be represented as isomorphic to a system of events, for it misses the property (iii) of Theorem 1. In other words, this K does not determine the structure of events underlying the experiment; the set of states for which we perform observations is not sufficient to determine events.

If $S=\{\alpha_1, \alpha_2\}$ is a two-element set, then S-probabilities are vectors in R^2 with coordinates in [0, 1]. Take as an example $K=\{(0, 0), (1, 0), (0, 1),$

$(1, 1)\}$. This is also a classical system isomorphic to the Boolean algebra $A = \{\mathbf{0}, a, a', \mathbf{1}\}$. We could interpret this structure by thinking of a coin with two faces A and B, viewing α_1 and α_2 as the states in which A or respectively B is up, and identifying our probabilities with the probabilities of seeing nothing (event $\mathbf{0}$), of seeing A (event a), of seeing B (event a'), and of seeing anything (event $\mathbf{1}$).

As another example we could consider $S = \{\alpha_1, \alpha_2, \alpha_3\}$ and take

$$K = \{(0, 0, 0), (1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 1, 0), (1, 0, 1), (0, 1, 1), 1, 1, 1)\}$$

It is easy to verify that the conditions (i)–(iv) of Theorem 2 are satisfied and consequently we have a classical system. This example could be easily generalized to n states.

To have a simple finite example of a nonclassical system, consider $S = \{\alpha_1, \alpha_2\}$ and the system of S-probabilities defined by

$$K_\delta = \{(0, 0), (1, 0), (0, 1), (\delta, 1-\delta), (1-\delta, \delta), (1, 1)\}$$

with any $0 < \delta < 1/2$. It is an easy exercise to verify that the system K_δ satisfies conditions (i)–(iii) of Theorems 1 and 2, but not condition (iv) of the latter, so it is a nonclassical system of S-probabilities, still representable as an event system. The corresponding orthocomplemented partially ordered set $\mathrm{L} = (L, \leq, ')$ is isomorphic to the lattice $\mathrm{M02} = \{\mathbf{0}, a, a', b, b', \mathbf{1}\}$ (there are no order relations between a and b, as well as between a and b'). This is the simplest orthomodular lattice which is not a Boolean algebra.

We can give a physical interpretation of this situation by considering, for instance, the polarization of a photon and viewing α_1 and α_2 as the states of linear polarization along two orthogonal axes, say x and y (with the photon propagating in the z direction). The S-probability $(1, 0)$ could then be interpreted as coming from the probability of transmission of the photon through a polarizer oriented along the x axis (and perpendicular to the z axis): indeed, the probability of transmission is 1 if the state is α_1 and 0 if the state is α_2. Similarly, the S-probability $(0, 1)$ would come from the transmission through the polarizer oriented along the y axis. The S-probability $(\delta, 1-\delta)$ would correspond to the transmission probabilities through a polarizer oriented along some new axis x' in the (x, y) plane: indeed, according to the Malus law, if γ is the angle between x and x', the transmission probability is $\delta = \cos^2 \gamma$ for the state α_1 and $\cos^2(\pi/2 - \gamma) = 1 - \delta$ for the state α_2. Similarly, the S-probability $(1-\delta, \delta)$ would correspond to the transmission probabilities through the polarizer oriented perpendicular to the x' axis. Should the angle γ be $\pi/4$, we would have $\delta = 1/2$ and the S-probabilities $(\delta, 1-\delta)$ and $(1-\delta, \delta)$ would coincide, thus losing their capability of defining unambiguously a positioning of the polarizer. The S-probabilities (0, 0) and (1, 1) correspond as usual to observing nothing and observing anything in both states.

Another obvious example of a nonclassical system of S-probabilities is obtained by taking for S the unit sphere of a separable Hilbert space H. For every orthogonal projection P on H we define $f_P(\alpha) = (P\alpha, \alpha)$ for all $a \in S$, and the set $K = \{f_P : P \text{ a projection}\}$ is a nonclassical system of S-probabilities. This system is isomorphic to L(H), the lattice of closed subsets of the Hilbert space H, and is interpreted as the logic of the Hilbert space quantum mechanics.

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